4.

Vector equation of a line in the x–y plane

- Interception/collision
- Vector equation of a straight line in the x–y plane
- Point of intersection of two straight lines
- Scalar product form of the equation of a straight line in the x–y plane
- Vector equations of curves in the x–y plane
- Vector equations of circles in the x–y plane
- Closest approach
- Distance from a point to a line
- Miscellaneous exercise four

Note

In this chapter we are considering two-dimensional vectors and lines, hence the mention of 'in the *x*–*y* plane' in the chapter title. The next chapter will consider similar concepts in three-dimensional space.

Situation

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Two ships, A and B, have the following position vectors, **r**, and velocity vectors, **v**, at 7 a.m. one morning.

$\mathbf{r}_A = (186\mathbf{i} + 44\mathbf{j})$ km. $\qquad \mathbf{v}_A = (8\mathbf{i} + 12\mathbf{j})$ km/h.

$\mathbf{r}_B = (228\mathbf{i} + 58\mathbf{j})$ km. $\qquad \mathbf{v}_B = (-16\mathbf{i} + 4\mathbf{j})$ km/h.

At the same time a third vessel, C, has position and velocity vectors as follows.

$\mathbf{r}_C = (215\mathbf{i} + 101\mathbf{j})$ km. $\qquad \mathbf{v}_C = (-12\mathbf{i} - 16\mathbf{j})$ km/h.

Prove that if A and B continue with their stated velocities they will collide and find the time of collision and the position vector of its location.

At the moment of impact between A and B, vessel C, responding to a distress call, immediately changes direction and heads to the scene of the collision at one-and-a-half times its previous speed. How many minutes after the collision occurred does vessel C arrive on the scene?

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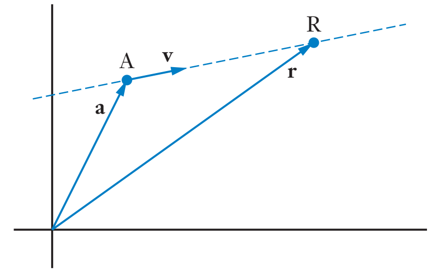
Interception/collision



Suppose that an object is at a point A, position vector **a**, and is moving along the path shown dashed in the diagram, with velocity **v**.

After a further t units of time the object will be at a point R, position vector **r**, where

$$\mathbf{r} = \mathbf{a} + t\mathbf{v} \qquad [1]$$

r, the position vector of the object at time t, is a function of time. Substituting a particular value of t into equation [1] will give the position vector of the object at that time. Being a function of time we could write the position vector as $\mathbf{r}(t)$.

Did you use this idea to solve the situation on the previous page? Example 1 is similar to that situation and the method of solution uses the above idea. (This approach is not the only way of solving the situation and you may well have used a different method.)

EXAMPLE 1

At noon, boats P and Q have position vectors (**r**) and velocity vectors (**v**) as follows:

$\mathbf{r}_P = -85\mathbf{i}$ km. $\qquad \mathbf{v}_P = (6\mathbf{i} + 8\mathbf{j})$ km/h.

$\mathbf{r}_Q = (-95\mathbf{i} + 40\mathbf{j})$ km. $\qquad \mathbf{v}_Q = (10\mathbf{i} - 8\mathbf{j})$ km/h.

a Prove that if P and Q continue with these velocities they will collide and find the time of collision and the position vector of its location.

b How far from the scene of the collision is the nearest coastal heliport, position vector $-56\mathbf{i} + 68\mathbf{j}$, the helicopter base from which help will arrive?

Solution

a Position vector of P, t hours past noon:
$$\mathbf{r}_P(t) = -85\mathbf{i} + t(6\mathbf{i} + 8\mathbf{j}) = (6t - 85)\mathbf{i} + 8t\mathbf{j}$$

Position vector of Q, t hours past noon:
$$\mathbf{r}_Q(t) = -95\mathbf{i} + 40\mathbf{j} + t(10\mathbf{i} - 8\mathbf{j}) = (10t - 95)\mathbf{i} + (40 - 8t)\mathbf{j}$$

The ships will collide if, for some value of t, $\mathbf{r}_P(t) = \mathbf{r}_Q(t)$

i.e. $(6t - 85)\mathbf{i} + 8t\mathbf{j} = (10t - 95)\mathbf{i} + (40 - 8t)\mathbf{j}$

Equating the **i** components $\qquad 6t - 85 = 10t - 95$

$t = 2.5$

Equating the **j** components $\qquad 8t = 40 - 8t$

$t = 2.5$

Thus when $t = 2.5$ the position vectors of P and Q have the same **i** components *and* the same **j** components. i.e. P and Q are in the same place at 2.30 p.m.

At this time $\mathbf{r}_P = \mathbf{r}_Q = -70\mathbf{i} + 20\mathbf{j}$

P and Q will collide at 2.30 p.m. at position vector $(-70\mathbf{i} + 20\mathbf{j})$ km.

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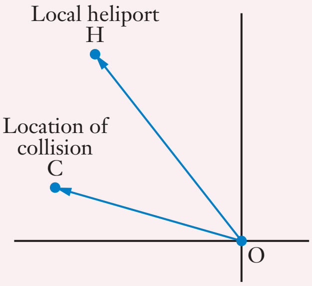
b The vector from the location of the collision, to the heliport will be $\overrightarrow{CH}$ (see diagram).

$$\begin{aligned}\overrightarrow{CH} &= \overrightarrow{CO} + \overrightarrow{OH}\\ &= -(-70\mathbf{i} + 20\mathbf{j}) + (-56\mathbf{i} + 68\mathbf{j})\\ &= 14\mathbf{i} + 48\mathbf{j}\\ \therefore\ |\overrightarrow{CH}| &= \sqrt{14^2 + 48^2}\text{ km}\\ &= 50\text{ km.}\end{aligned}$$



The scene of the collision is 50 km from the coastal heliport.

In the *Preliminary work* it was mentioned that the vector $a\mathbf{i} + b\mathbf{j}$ is sometimes written as the ordered pair $\langle a, b\rangle$ and sometimes as a column matrix $\begin{pmatrix} a \\ b \end{pmatrix}$.

The latter notation is used in the working of the next example.

EXAMPLE 2

The position vectors ($\mathbf{r}$) and velocity vectors ($\mathbf{v}$) of two ships A and B at certain times on a particular day were as follows:

At 9.00 a.m. $\mathbf{r}_A = (30\mathbf{i} + 50\mathbf{j})$ km. $\mathbf{v}_A = (12\mathbf{i} - 3\mathbf{j})$ km/h.

At 9.30 a.m. $\mathbf{r}_B = (48\mathbf{i} + 30\mathbf{j})$ km. $\mathbf{v}_B = (8\mathbf{i} + 2\mathbf{j})$ km/h.

Show that if the two ships continue with these velocity vectors they will not collide.

Solution

At t hours past 9 a.m.

$$\begin{aligned}\mathbf{r}_A(t) &= \begin{pmatrix} 30 \\ 50 \end{pmatrix} + t\begin{pmatrix} 12 \\ -3 \end{pmatrix}\\ &= \begin{pmatrix} 30 + 12t \\ 50 - 3t \end{pmatrix}\\ \mathbf{r}_B(t) &= \begin{pmatrix} 48 \\ 30 \end{pmatrix} + (t - 0.5)\begin{pmatrix} 8 \\ 2 \end{pmatrix} \qquad (t \geq 0.5)\\ &= \begin{pmatrix} 44 + 8t \\ 29 + 2t \end{pmatrix}\end{aligned}$$

Position vectors of A and B will have same $\mathbf{i}$ component when $30 + 12t = 44 + 8t$

i.e. when $t = 3.5$

Position vectors of A and B will have same $\mathbf{j}$ component when $50 - 3t = 29 + 2t$

i.e. when $t = 4.2$

The same $\mathbf{i}$ component of position vector occurs at 12.30 p.m. but the same $\mathbf{j}$ component occurs at 1.12 p.m. Thus A and B do not collide.

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Exercise 4A

1 At the times stated, the position vectors ($\mathbf{r}$) and velocity vectors ($\mathbf{v}$) of bodies A, B, C, D, E and F are as follows:

$\mathbf{r}_A$	=	$(5\mathbf{i} + 4\mathbf{j})$ km	$\mathbf{v}_A$	=	$(10\mathbf{i} - \mathbf{j})$ km/h	at	8 a.m.
$\mathbf{r}_B$	=	$(6\mathbf{i} - 8\mathbf{j})$ km	$\mathbf{v}_B$	=	$(2\mathbf{i} + 8\mathbf{j})$ km/h	at	8 a.m.
$\mathbf{r}_C$	=	$(2\mathbf{i} + 3\mathbf{j})$ km	$\mathbf{v}_C$	=	$(-4\mathbf{i} + 3\mathbf{j})$ km/h	at	8 a.m.
$\mathbf{r}_D$	=	$(9\mathbf{i} - 10\mathbf{j})$ km	$\mathbf{v}_D$	=	$(10\mathbf{i} + 6\mathbf{j})$ km/h	at	7 a.m.
$\mathbf{r}_E$	=	$(16\mathbf{i} + 7\mathbf{j})$ km	$\mathbf{v}_E$	=	$(-4\mathbf{i} + 3\mathbf{j})$ km/h	at	9 a.m.
$\mathbf{r}_F$	=	$(2\mathbf{i} + 3\mathbf{j})$ km	$\mathbf{v}_F$	=	$(12\mathbf{i} - 8\mathbf{j})$ km/h	at	8.30 a.m.

In each case write, in the form $f(t)\mathbf{i} + g(t)\mathbf{j}$, an expression for the position vector of the body t hours after 8 a.m. (Assume each velocity continues unchanged.)

2 A ship has a position vector $(7\mathbf{i} + 10\mathbf{j})$ km at 5 a.m. and is moving with velocity vector $(3\mathbf{i} + 4\mathbf{j})$ km/h. If the ship continues with this velocity what will be its position vector at

a 6 a.m.? **b** 7 a.m.? **c** 9 a.m.?

d What is the speed of the ship?

e At 8 a.m, how far is the ship from a lighthouse, position vector $(21\mathbf{i} + 20\mathbf{j})$ km?

3 With respect to an origin O a boat has position vector $(9\mathbf{i} + 36\mathbf{j})$ km at 10 a.m. and is moving with velocity $(2\mathbf{i} + 12\mathbf{j})$ km/h. Assuming that the boat had been travelling at that velocity for some hours what was its position vector at 9 a.m.?
How far was the boat from O at

a 9 a.m.? **b** 8 a.m.?

4 At 3 p.m. one day two ships A and B have position vectors, $\mathbf{r}$ km, and velocity vectors, $\mathbf{v}$ km/h, as follows:

$$\mathbf{r}_A = 21\mathbf{i} + 7\mathbf{j} \qquad \mathbf{v}_A = 10\mathbf{i} + 5\mathbf{j}$$
$$\mathbf{r}_B = 25\mathbf{i} - 6\mathbf{j} \qquad \mathbf{v}_B = 7\mathbf{i} + 10\mathbf{j}$$

Assuming both ships maintain these velocities, how far apart will the ships be at

a 3 p.m.? **b** 4 p.m.? **c** 5 p.m.?

5 At 8 a.m. one morning two ships A and B have position vectors, $\mathbf{r}$ km, and velocity vectors, $\mathbf{v}$ km/h, as follows:

$$\mathbf{r}_A = -5\mathbf{i} + 13\mathbf{j} \qquad \mathbf{v}_A = 7\mathbf{i} - 2\mathbf{j}$$
$$\mathbf{r}_B = -3\mathbf{j} \qquad \mathbf{v}_B = -3\mathbf{i} + 2\mathbf{j}$$

Assuming both ships maintain these velocities, how far apart will the ships be at

a 9 a.m.? **b** 10 a.m.?

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6 At 8 a.m. one morning two ships A and B have position vectors, $\mathbf{r}$ km, and velocity vectors, $\mathbf{v}$ km/h, as follows:

$$\mathbf{r}_A = 28\mathbf{i} - 5\mathbf{j} \qquad \mathbf{v}_A = -8\mathbf{i} + 4\mathbf{j}$$
$$\mathbf{r}_B = 24\mathbf{j} \qquad \mathbf{v}_B = 6\mathbf{i} + 2\mathbf{j}$$

a Obtain an expression in the form $f(t)\mathbf{i} + g(t)\mathbf{j}$ for the position vector of each ship t hours after 8 a.m.

b When will the ships be 25 km apart?

For each of the following determine whether the ships A and B will collide if they continue with the velocities given below. For those that do collide, find when this occurs and the position vector of the scene of the collision.

7	$\mathbf{r}_A = (12\mathbf{i} + 61\mathbf{j})$ km	$\mathbf{v}_A = (7\mathbf{i} - 8\mathbf{j})$ km/h	at	8.00 a.m.
	$\mathbf{r}_B = (57\mathbf{i} - 29\mathbf{j})$ km	$\mathbf{v}_B = (-2\mathbf{i} + 10\mathbf{j})$ km/h	at	8.00 a.m.
8	$\mathbf{r}_A = (-11\mathbf{i} - 8\mathbf{j})$ km	$\mathbf{v}_A = (7\mathbf{i} - \mathbf{j})$ km/h	at	8.00 a.m.
	$\mathbf{r}_B = (-2\mathbf{i} - 4\mathbf{j})$ km	$\mathbf{v}_B = (4\mathbf{i} + 5\mathbf{j})$ km/h	at	8.00 a.m.
9	$\mathbf{r}_A = (24\mathbf{i} - 25\mathbf{j})$ km	$\mathbf{v}_A = (-3\mathbf{i} + 4\mathbf{j})$ km/h	at	8.00 a.m.
	$\mathbf{r}_B = (-9\mathbf{i} + 33\mathbf{j})$ km	$\mathbf{v}_B = (2\mathbf{i} - 5\mathbf{j})$ km/h	at	9.00 a.m.
10	$\mathbf{r}_A = (-6\mathbf{i} + 44\mathbf{j})$ km	$\mathbf{v}_A = (4\mathbf{i} - 6\mathbf{j})$ km/h	at	9.30 a.m.
	$\mathbf{r}_B = (2\mathbf{i} - 18\mathbf{j})$ km	$\mathbf{v}_B = (2\mathbf{i} + 7\mathbf{j})$ km/h	at	9.00 a.m.
11	$\mathbf{r}_A = (-11\mathbf{i} + 4\mathbf{j})$ km	$\mathbf{v}_A = (10\mathbf{i} - 4\mathbf{j})$ km/h	at	noon.
	$\mathbf{r}_B = (3\mathbf{i} - 5\mathbf{j})$ km	$\mathbf{v}_B = (7\mathbf{i} + 5\mathbf{j})$ km/h	at	12.30 p.m.

12 At 8 a.m. boats P, Q and R have position vectors ($\mathbf{r}$) and velocity vectors ($\mathbf{v}$) as follows:

$$\mathbf{r}_P = (-23\mathbf{i} + 3\mathbf{j}) \text{ km.} \qquad \mathbf{v}_P = (18\mathbf{i} + 4\mathbf{j}) \text{ km/h.}$$
$$\mathbf{r}_Q = (7\mathbf{i} + 30\mathbf{j}) \text{ km.} \qquad \mathbf{v}_Q = (12\mathbf{i} - 10\mathbf{j}) \text{ km/h.}$$
$$\mathbf{r}_R = (32\mathbf{i} - 30\mathbf{j}) \text{ km.} \qquad \mathbf{v}_R = (2\mathbf{i} + 14\mathbf{j}) \text{ km/h.}$$

a Prove that if the boats continue with these velocities two of them will collide, stating which two it will be, the time of collision and the position vector of its location.

b How far will the third boat be from the scene of the collision at the moment it occurs?

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Vector equation of a straight line in the x–y plane

Suppose a ship is at the point with position vector $(2\mathbf{i} + 5\mathbf{j})$ km and is moving with a constant velocity of $(4\mathbf{i} + \mathbf{j})$ km/h.

At a time t hours later the ship will be at the point with position vector $\mathbf{r}$ where

$$\mathbf{r} = (2\mathbf{i} + 5\mathbf{j}) + t(4\mathbf{i} + \mathbf{j}).$$

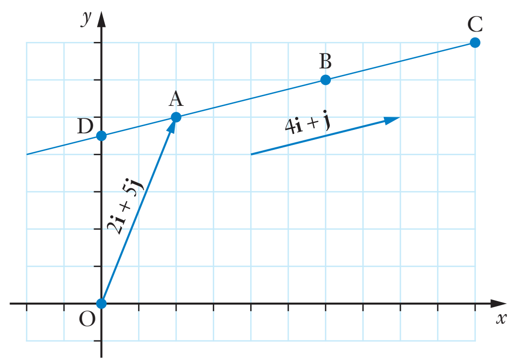
Let us now remove the context of a moving ship and simply consider the line through point A, position vector $2\mathbf{i} + 5\mathbf{j}$, and parallel to the vector $4\mathbf{i} + \mathbf{j}$.

Consider points B, C and D on this line and let point O be the origin (see diagram).

B has position vector $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = (2\mathbf{i} + 5\mathbf{j}) + 1(4\mathbf{i} + \mathbf{j}) = 6\mathbf{i} + 6\mathbf{j}$

C has position vector $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = (2\mathbf{i} + 5\mathbf{j}) + 2(4\mathbf{i} + \mathbf{j}) = 10\mathbf{i} + 7\mathbf{j}$

D has position vector $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = (2\mathbf{i} + 5\mathbf{j}) + (-0.5)(4\mathbf{i} + \mathbf{j}) = 4.5\mathbf{j}$



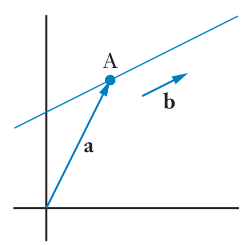
The position vector of every point on the line through A, position vector $2\mathbf{i} + 5\mathbf{j}$, and parallel to the vector $4\mathbf{i} + \mathbf{j}$ can be expressed in the form:

$2\mathbf{i} + 5\mathbf{j} + \lambda(4\mathbf{i} + \mathbf{j})$ for a suitable choice of the scalar, λ.

For point A, $\lambda = 0$, for B, $\lambda = 1$, for C, $\lambda = 2$, for D, $\lambda = -0.5$.

To generalise: Every point on the line through A, position vector $\mathbf{a}$, and parallel to the vector $\mathbf{b}$ has a position vector that can be expressed in the form:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$



The equation: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

is the vector equation of the line parallel to $\mathbf{b}$ and through the point with position vector $\mathbf{a}$.

Every point on the line has a position vector that can be expressed in the form $\mathbf{a} + \lambda\mathbf{b}$ for some suitable value λ, and furthermore, any point not lying on the line cannot be expressed in this form.

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EXAMPLE 3

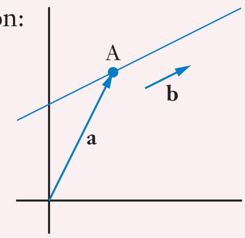
a Find the vector equation of the line passing through the point A, position vector $2\mathbf{i} + 3\mathbf{j}$, parallel to the vector $4\mathbf{i} - 6\mathbf{j}$.

b Determine whether or not each of the following points lie on the line:

B, position vector $10\mathbf{i} - 9\mathbf{j}$. C, position vector $8\mathbf{i} - 5\mathbf{j}$.

Solution

a The line through A, position vector $\mathbf{a}$, and parallel to $\mathbf{b}$ has vector equation:



$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}.$$

Thus the line through A, position vector $2\mathbf{i} + 3\mathbf{j}$, and parallel to $4\mathbf{i} - 6\mathbf{j}$ has vector equation

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(4\mathbf{i} - 6\mathbf{j}).$$

i.e. $$\mathbf{r} = (2 + 4\lambda)\mathbf{i} + (3 - 6\lambda)\mathbf{j}$$

b If B, position vector $10\mathbf{i} - 9\mathbf{j}$, lies on $\mathbf{r} = (2 + 4\lambda)\mathbf{i} + (3 - 6\lambda)\mathbf{j}$
there must exist some λ for which $10\mathbf{i} - 9\mathbf{j} = (2 + 4\lambda)\mathbf{i} + (3 - 6\lambda)\mathbf{j}$

i.e. $10 = 2 + 4\lambda$ and $-9 = 3 - 6\lambda$
$\lambda = 2$ and $\lambda = 2$

Thus a suitable value of λ does exist.
Point B *does* lie on $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(4\mathbf{i} - 6\mathbf{j})$.

If C, position vector $8\mathbf{i} - 5\mathbf{j}$, lies on $\mathbf{r} = (2 + 4\lambda)\mathbf{i} + (3 - 6\lambda)\mathbf{j}$
there must exist some λ for which $8\mathbf{i} - 5\mathbf{j} = (2 + 4\lambda)\mathbf{i} + (3 - 6\lambda)\mathbf{j}$

i.e. $8 = 2 + 4\lambda$ and $-5 = 3 - 6\lambda$
$\lambda = \frac{3}{2}$ and $\lambda = \frac{4}{3}$

Point C does *not* lie on $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(4\mathbf{i} - 6\mathbf{j})$.

It is important to realise that with the usual convention of $\mathbf{i}$ being a unit vector in the positive x direction and $\mathbf{j}$ a unit vector in the positive y direction, there is no conflict between:

- the vector equation of a straight line, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

and

- the cartesian equation of a straight line, $y = mx + c$.

The position vector, $\mathbf{r}$, of any point on the line will obey the rule $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.

The cartesian coordinates (x, y) of any point on the line will obey $y = mx + c$.

This consistency between the vector equation of a line and the cartesian equation of the same line is demonstrated on the next page.

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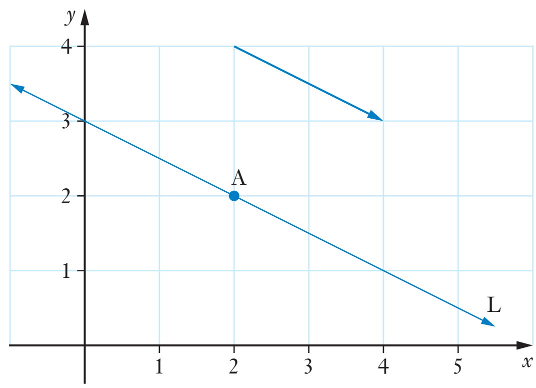
Consider the line L shown on the right.

This line passes through A, position vector $2\mathbf{i} + 2\mathbf{j}$ and is parallel to the vector $2\mathbf{i} - \mathbf{j}$.

Thus the vector equation of the line can be written:

$$\begin{aligned}\mathbf{r} &= 2\mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j}) \\ &= (2 + 2\lambda)\mathbf{i} + (2 - \lambda)\mathbf{j}\end{aligned}$$

Now consider some general point P, cartesian coordinates (x, y), lying on this line. The position vector of P will be $x\mathbf{i} + y\mathbf{j}$.



But point P lies on the line and so $\quad x\mathbf{i} + y\mathbf{j} = (2 + 2\lambda)\mathbf{i} + (2 - \lambda)\mathbf{j}$

Thus $\quad x = 2 + 2\lambda \qquad$ [1]

and $\quad y = 2 - \lambda \qquad$ [2]

Eliminating λ from [1] and [2] gives $\quad y = -\dfrac{1}{2}x + 3.$

This equation is exactly as we would expect for a line passing through (0, 3) and with gradient –0.5.

Thus, as stated, there is no conflict between the vector equation $\quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

and the cartesian equation $\quad y = mx + c.$

Note

- You may find it convenient for this work to write the vector $a\mathbf{i} + b\mathbf{j}$ in the matrix form $\begin{pmatrix} a \\ b \end{pmatrix}$.
- Equations [1] and [2] above are called the **parametric equations** of the line and λ is called the **parameter**. It acts as a sort of 'go-between' linking x and y. It is the 'interpreter' through which they relate to each other.
- Letters other than λ may be used for the parameter. Indeed if time is involved it makes sense to use t rather than λ.
- We said that line L passed through A, position vector $2\mathbf{i} + 2\mathbf{j}$ and was parallel to $2\mathbf{i} - \mathbf{j}$ and wrote the vector equation as:

  $$\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j}). \qquad [3]$$

  We could equally well have said that the line passed through the point with position vector $4\mathbf{i} + \mathbf{j}$ and was parallel to $2\mathbf{i} - \mathbf{j}$. We would then have written the vector equation as:

  $$\mathbf{r} = 4\mathbf{i} + \mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j}). \qquad [4]$$

  Though these equations may appear different both [3] and [4] define the same line. The position vector obtained by substituting a specific value of λ into [3] would be obtained by substituting $(\lambda - 1)$ into [4].

  e.g. Substituting $\lambda = 2$ into equation [3] gives the position vector $6\mathbf{i}$. This same position vector is obtained when $\lambda = 1$ is substituted into [4].

  Thus in some questions, when comparing your vector equation of a line with the one given in the answers, do not automatically assume that your answer, because it looks different, is wrong. It may simply be a different way, and equally acceptable way, of defining the same line.

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EXAMPLE 4

Find **a** the vector equation, **b** the parametric equations and **c** the cartesian equation,

of the line through the point with position vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$, parallel to $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

Solution

a The vector equation is $\quad \mathbf{r} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad$ i.e $\quad \mathbf{r} = \begin{pmatrix} 3+2\lambda \\ -4+5\lambda \end{pmatrix}$.

b Considering the general point, position vector $\begin{pmatrix} x \\ y \end{pmatrix}$: $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3+2\lambda \\ -4+5\lambda \end{pmatrix}$

Thus the parametric equations are $\begin{cases} x = 3+2\lambda \\ y = -4+5\lambda \end{cases}$

c Eliminating λ gives: $\quad x = 3 + 2\left(\dfrac{y+4}{5}\right)$

i.e. $\quad 2y = 5x - 23$.

The cartesian equation is $2y = 5x - 23$.

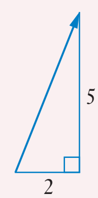
Alternatively the cartesian equation could be found as follows:

If the line is parallel to $2\mathbf{i} + 5\mathbf{j}$ it must have gradient $\dfrac{5}{2}$.

Therefore the equation is of the form $\quad y = 2.5x + c$

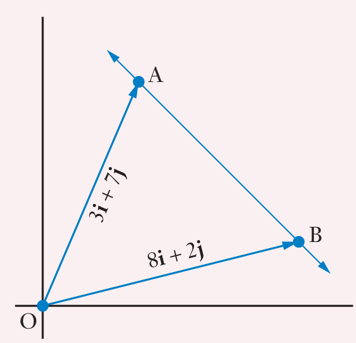
But line passes through $(3, -4)$, thus $\quad -4 = 2.5(3) + c \quad$ i.e. $\quad c = -11.5$

The cartesian equation is $y = 2.5x - 11.5$.



EXAMPLE 5

Find the vector equation of the line passing through the point A, position vector $3\mathbf{i} + 7\mathbf{j}$, and B, position vector $8\mathbf{i} + 2\mathbf{j}$.



Solution

The line is parallel to $\overrightarrow{AB}$.

$$\begin{aligned} \text{Now} \quad \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -(3\mathbf{i} + 7\mathbf{j}) + (8\mathbf{i} + 2\mathbf{j}) \\ &= 5\mathbf{i} - 5\mathbf{j} \end{aligned}$$

The line is parallel to $5\mathbf{i} - 5\mathbf{j}$ and passes through point A, position vector $3\mathbf{i} + 7\mathbf{j}$. Thus the vector equation of the line is $\quad \mathbf{r} = 3\mathbf{i} + 7\mathbf{j} + \lambda(5\mathbf{i} - 5\mathbf{j})$.

i.e. $\quad \mathbf{r} = (3 + 5\lambda)\mathbf{i} + (7 - 5\lambda)\mathbf{j}$

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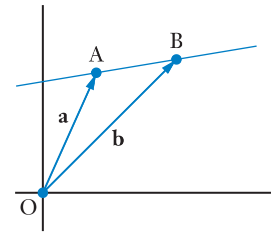
In the previous example, instead of being given a vector parallel to the line and the position vector of a point on the line, we were given the position vectors of two points lying on the line.

The general situation is shown on the right with points A and B lying on the line, position vectors **a** and **b** respectively.



The line is parallel to $\overrightarrow{AB}$, and

$$\overrightarrow{AB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$

Thus the line passes through point A, position vector **a**, and is parallel to **b** – **a**.

The vector equation of the line through points A and B, position vectors **a** and **b** respectively is therefore:

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

The answer to the previous example could have been obtained by direct substitution of $\mathbf{a} = 3\mathbf{i} + 7\mathbf{j}$ and $\mathbf{b} = 8\mathbf{i} + 2\mathbf{j}$ into the above formula.

The equation: $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

is the vector equation of the line through points A and B, position vectors **a** and **b** respectively.

Exercise 4B

For questions **1** to **6** find the vector equation of the line parallel to **b** and through the point with position vector **a**.

1 $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} - \mathbf{j}$.

2 $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$.

3 $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -2\mathbf{j}$.

4 $\mathbf{a} = 5\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - 10\mathbf{j}$.

5 $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

6 $\mathbf{a} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

For questions **7** to **12** find the vector equation of the line passing through the point with position vector **a** and the point with position vector **b**.

7 $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$.

8 $\mathbf{a} = 6\mathbf{i} + 7\mathbf{j}$, $\mathbf{b} = -5\mathbf{i} + 2\mathbf{j}$.

9 $\mathbf{a} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

10 $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

11 $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -1 \\ 9 \end{pmatrix}$

12 $\mathbf{a} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$

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13 Points A, B and C lie on the line $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} - 4\mathbf{j})$ and have position vectors given by $\lambda = -1$, $\lambda = 1$ and $\lambda = 2$ respectively.

Find **a** $\overrightarrow{AB}$, **b** $|\overrightarrow{BC}|$, **c** $\overrightarrow{AB} : \overrightarrow{BC}$.

14 Find **a** the vector equation, **b** the parametric equations and **c** the cartesian equation of the line passing through the point A, position vector $5\mathbf{i} - \mathbf{j}$, and parallel to $7\mathbf{i} + 2\mathbf{j}$.

15 Find **a** the vector equation, **b** the parametric equations and **c** the cartesian equation of the line passing through the point A, position vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, and parallel to $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

16 Find **a** the vector equation, **b** the parametric equations and **c** the cartesian equation of the line passing through point A, position vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$, and parallel to $\begin{pmatrix} 7 \\ -8 \end{pmatrix}$.

17 Find **a** the vector equation and **b** the cartesian equation of the line with parametric equations $x = 2 - 3\lambda$, $y = -5 + 2\lambda$.

18 Points D, E and F lie on the line $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and have position vectors given by $\lambda = -1$, $\lambda = 2$ and $\lambda = 3$ respectively.

Find **a** $\overrightarrow{EF}$, **b** $\overrightarrow{ED}$, **c** $|\overrightarrow{DE}|$,

d $\overrightarrow{DE} : \overrightarrow{EF}$, **e** $\overrightarrow{DE} : \overrightarrow{FE}$, **f** $|\overrightarrow{DE}| : |\overrightarrow{FE}|$.

19 Find the vector equation of the line passing through the point A, which has position vector $7\mathbf{i} - 2\mathbf{j}$, and which is parallel to the vector $-2\mathbf{i} + 6\mathbf{j}$.

Determine whether each of the following points lie on the line.

B, position vector $\mathbf{i} + 16\mathbf{j}$. C, position vector $2\mathbf{i} + 13\mathbf{j}$.

D, position vector $8\mathbf{i} - 7\mathbf{j}$. E, position vector $-2\mathbf{i} + 5\mathbf{j}$.

20 Find the vector equation of the line passing through the point F, position vector $\begin{pmatrix} 4 \\ -9 \end{pmatrix}$, and which is parallel to the vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Determine whether each of the following points lie on the line.

G, position vector $\begin{pmatrix} 5 \\ 9 \end{pmatrix}$. H, position vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$. I, position vector $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$.

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21 All of the points A to F, with position vectors as given below, lie on the line

$$\mathbf{r} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 8 \end{pmatrix}.$$

A, position vector $-3\mathbf{i} + a\mathbf{j}$.

B, position vector $b\mathbf{i} + 23\mathbf{j}$.

C, position vector $<-9, c>$.

D, position vector $<d, -21>$.

E, position vector $\begin{pmatrix} 12 \\ e \end{pmatrix}$.

F, position vector $\begin{pmatrix} f \\ f \end{pmatrix}$.

Determine the values of a, b, c, d, e and f.

22 Find the vector equation of the line passing through the point with position vector $5\mathbf{i} - 6\mathbf{j}$ and parallel to the line $\mathbf{r} = (2 + \lambda)\mathbf{i} + (3 - \lambda)\mathbf{j}$.

23 Find the vector equation of the line passing through the point with position vector $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and parallel to the line $\mathbf{r} = \begin{pmatrix} 2 + 3\lambda \\ 1 - 4\lambda \end{pmatrix}$.

24 A ship travels with constant velocity $(6\mathbf{i} - 10\mathbf{j})$ km/h and passes through the point with position vector $(2\mathbf{i} + 12\mathbf{j})$ km. Find the cartesian equation of the path of the ship.

25 The line $\mathbf{r} = 2\mathbf{i} + 8\mathbf{j} + \lambda(\mathbf{i} - 2\mathbf{j})$ cuts the x-axis at A and the y-axis at B. Find the position vectors of A and B.

26 The line $\mathbf{r} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ cuts the x-axis at A and passes through B, position vector $\begin{pmatrix} 11 \\ c \end{pmatrix}$.

Find the position vector of A and the value of c.

27 Points A, B, C and D are collinear and have position vectors $2\mathbf{i} + 3\mathbf{j}$, $b\mathbf{i} + 7\mathbf{j}$, $5\mathbf{i} - 4\mathbf{j}$, and $-2\mathbf{i} + d\mathbf{j}$ respectively. Find the vector equation of the line through the four points and the values of b and d.

28 The vector equations $\mathbf{r} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 9 \\ d \end{pmatrix} + \mu \begin{pmatrix} 2 \\ c \end{pmatrix}$ represent the same straight line.

Find the values of c and d.

29 The vector equations $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + \lambda(3\mathbf{i} + 4\mathbf{j})$ and $\mathbf{r} = e\mathbf{i} + 5\mathbf{j} + \mu(\mathbf{i} + f\mathbf{j})$ represent the same straight line. Find the values of e and f.

30 Three sets of parametric equations are given below. Which is the ‘odd set out’ and why?

[1] $\begin{cases} x = 1 + 2\lambda \\ y = \lambda + 3 \end{cases}$

[2] $\begin{cases} x = 2\lambda - 2 \\ y = 1 + \lambda \end{cases}$

[3] $\begin{cases} x = 8 + 2\lambda \\ y = 6 + \lambda \end{cases}$

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Use the concept of the **scalar product** of two vectors for each of the following questions. (See the *Preliminary work* section if you need to refresh your memory of this concept.)

31 Prove that the lines L_1 and L_2 are perpendicular given that:

L_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and L_2 has equation $\mathbf{r} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

32 Line L_1 has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the vector equation of the line perpendicular to L_1 and passing through the point A, position vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

33 Find to the nearest degree the acute angle between the lines L_1 and L_2 if L_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and L_2 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Point of intersection of two straight lines

Consider the two lines

$$L_1: \quad \mathbf{r} = \begin{pmatrix} 7 \\ 17 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -2 \end{pmatrix}, \qquad L_2: \quad \mathbf{r} = \begin{pmatrix} 16 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

The lines are straight, and not parallel. (How do we know they are not parallel?) Therefore they must intersect somewhere.

The point common to both lines will be such that

$$\begin{pmatrix} 7 \\ 17 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 16 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

i.e.

$$\begin{cases} 7 + 7\lambda = 16 + 3\mu \\ 17 - 2\lambda = 3 + 2\mu \end{cases}$$

Solving simultaneously gives $\lambda = 3$ and $\mu = 4$

With $\lambda = 3$ line L_1 gives $\mathbf{r} = \begin{pmatrix} 7 \\ 17 \end{pmatrix} + 3\begin{pmatrix} 7 \\ -2 \end{pmatrix}$ i.e. $\mathbf{r} = \begin{pmatrix} 28 \\ 11 \end{pmatrix}$.

With $\mu = 4$ line L_2 gives $\mathbf{r} = \begin{pmatrix} 16 \\ 3 \end{pmatrix} + 4\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ i.e. $\mathbf{r} = \begin{pmatrix} 28 \\ 11 \end{pmatrix}$.

Lines L_1 and L_2 intersect at the point with position vector $28\mathbf{i} + 11\mathbf{j}$.

The next two examples show this same approach applied to determining whether two moving objects collide, a situation encountered at the beginning of this chapter.

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EXAMPLE 6

At time $t = 0$ seconds the position vectors ($\mathbf{r}$ m) and velocity vectors ($\mathbf{v}$ m/s) of two particles A and B are as follows:

$$\mathbf{r}_A = 4\mathbf{i} + 20\mathbf{j} \qquad \mathbf{v}_A = \mathbf{i} - \mathbf{j}$$
$$\mathbf{r}_B = 5\mathbf{i} + 4\mathbf{j} \qquad \mathbf{v}_B = \mathbf{i} + 2\mathbf{j}$$

Determine whether, in the subsequent motion, the paths of the particles cross (or meet). If they do, determine the position vector of this point and determine if a collision between the particles is involved.

Solution

At time t_1, $t_1 > 0$, particle A will have position vector $\quad \mathbf{r} = \begin{pmatrix} 4 \\ 20 \end{pmatrix} + t_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

At time t_2, $t_2 > 0$, particle B will have position vector $\quad \mathbf{r} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + t_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

For these position vectors to be equal: $\quad \begin{pmatrix} 4 \\ 20 \end{pmatrix} + t_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + t_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

i.e. $\quad \begin{cases} 4 + t_1 = 5 + t_2 \\ 20 - t_1 = 4 + 2t_2 \end{cases}$

Solving simultaneously gives $t_1 = 6$ and $t_2 = 5$.

With $t_1 = 6$ particle A has position vector $\quad \mathbf{r} = \begin{pmatrix} 4 \\ 20 \end{pmatrix} + 6\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad$ i.e. $\quad \mathbf{r} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$.

With $t_2 = 5$ particle B has position vector $\quad \mathbf{r} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad$ i.e. $\quad \mathbf{r} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$.

Thus particles A and B each pass through the point with position vector $10\mathbf{i} + 14\mathbf{j}$, but at different times, both greater than zero.

Thus in the subsequent motion the paths of the particles cross at the point with position vector $10\mathbf{i} + 14\mathbf{j}$ but a collision is not involved.

EXAMPLE 7

At time $t = 0$ seconds the position vectors ($\mathbf{r}$ m) and velocity vectors ($\mathbf{v}$ m/s) of two particles A and B are as given below:

$$\mathbf{r}_A = 9\mathbf{i} + 55\mathbf{j} \qquad \mathbf{v}_A = -\mathbf{i} - 3\mathbf{j}$$
$$\mathbf{r}_B = 24\mathbf{i} - 5\mathbf{j} \qquad \mathbf{v}_B = -2\mathbf{i} + \mathbf{j}$$

Determine whether, in the subsequent motion, the paths of the particles cross (or meet). If they do, determine the position vector of this point and determine if a collision between the particles is involved.

Solution

At time t_1, $t_1 > 0$, particle A will have position vector $\quad \mathbf{r} = \begin{pmatrix} 9 \\ 55 \end{pmatrix} + t_1\begin{pmatrix} -1 \\ -3 \end{pmatrix}$.

At time t_2, $t_2 > 0$, particle B will have position vector $\quad \mathbf{r} = \begin{pmatrix} 24 \\ -5 \end{pmatrix} + t_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

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For these position vectors to be equal: $\begin{pmatrix} 9 \\ 55 \end{pmatrix} + t_1\begin{pmatrix} -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 24 \\ -5 \end{pmatrix} + t_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

i.e., $\begin{cases} 9 - t_1 = 24 - 2t_2 \\ 55 - 3t_1 = -5 + t_2 \end{cases}$

Solving simultaneously gives $t_1 = 15$ and $t_2 = 15$.

In the subsequent motion, i.e. $t > 0$, the paths of the particles meet.

With $t_1 = 15$ particle A has position vector $\mathbf{r} = \begin{pmatrix} 9 \\ 55 \end{pmatrix} + 15\begin{pmatrix} -1 \\ -3 \end{pmatrix}$ i.e. $\mathbf{r} = \begin{pmatrix} -6 \\ 10 \end{pmatrix}$.

With $t_2 = 15$ particle B has position vector $\mathbf{r} = \begin{pmatrix} 24 \\ -5 \end{pmatrix} + 15\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ i.e. $\mathbf{r} = \begin{pmatrix} -6 \\ 10 \end{pmatrix}$.

Thus particles A and B are each at the point with position vector $-6\mathbf{i} + 10\mathbf{j}$ at time $t = 15$ seconds. A collision is involved.

Exercise 4C

1 Find the position vector of the point of intersection of lines L_1 and L_2, vector equations,

L_1: $\mathbf{r} = 14\mathbf{i} - \mathbf{j} + \lambda(5\mathbf{i} - 4\mathbf{j})$ L_2: $\mathbf{r} = 9\mathbf{i} - 4\mathbf{j} + \mu(-4\mathbf{i} + 6\mathbf{j})$

2 Find the position vector of the point of intersection of lines L_1 and L_2, vector equations,

L_1: $\mathbf{r} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, L_2: $\mathbf{r} = \begin{pmatrix} -10 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

3 Find the position vector of the point of intersection of lines L_1 and L_2, vector equations,

L_1: $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -10 \end{pmatrix}$, L_2: $\mathbf{r} = \begin{pmatrix} -5 \\ -9 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 7 \end{pmatrix}$.

For each of questions **4**, **5** and **6** the given information shows the position vectors ($\mathbf{r}$ m) and velocity vectors ($\mathbf{v}$ m/s) of two particles A and B at time $t = 0$ seconds.

In each case determine whether, in the subsequent motion, the paths of the particles cross (or meet), and, if they do, determine the position vector of this point and determine if a collision between the particles is involved, explaining your answer.

4 $\mathbf{r}_A = 16\mathbf{i}$ $\mathbf{v}_A = 3\mathbf{i} + 2\mathbf{j}$

$\mathbf{r}_B = -\mathbf{i} + 6\mathbf{j}$ $\mathbf{v}_B = 2\mathbf{i} - 3\mathbf{j}$

5 $\mathbf{r}_A = \mathbf{i} + 4\mathbf{j}$ $\mathbf{v}_A = 4\mathbf{i} + 1\mathbf{j}$

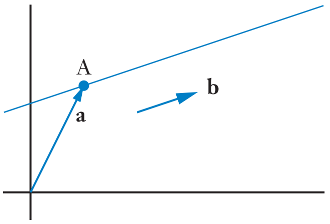
$\mathbf{r}_B = 37\mathbf{i} - 20\mathbf{j}$ $\mathbf{v}_B = -2\mathbf{i} + 5\mathbf{j}$

6 $\mathbf{r}_A = \mathbf{i} + 19\mathbf{j}$ $\mathbf{v}_A = 2\mathbf{i} - \mathbf{j}$

$\mathbf{r}_B = 3\mathbf{i} + 8\mathbf{j}$ $\mathbf{v}_B = 3\mathbf{i} + \mathbf{j}$

Scalar product form of the equation of a straight line in the x–y plane

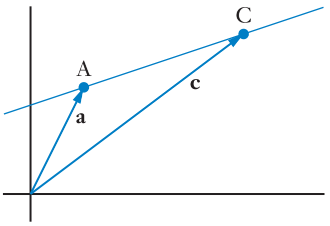
- If we are given vector **a**, the position vector of one point lying on a straight line, and vector **b**, a vector parallel to the line, then the line is uniquely defined. This enabled us to write the equation of the line as



$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}.$$

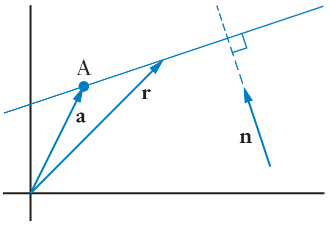
- If we are given the position vectors of two points lying on the line, the line is again uniquely defined.

  This enabled us to write the equation of the line in the form



$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{c} - \mathbf{a}).$$

- A straight line in the **i**–**j** plane (or x–y plane) may also be uniquely defined by stating vector **a**, the position vector of one point lying on the line, and vector **n**, a vector in the **i**–**j** plane and perpendicular to the line.



  If **r** is the position vector of a general point lying on the line then $(\mathbf{r} - \mathbf{a})$ is parallel to the line.

  Thus $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$

  i.e. $\mathbf{r} \cdot \mathbf{n} - \mathbf{a} \cdot \mathbf{n} = 0$

  giving $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

  With **a** and **n** known, $\mathbf{a} \cdot \mathbf{n}$ is a constant and the equation can be written $\mathbf{r} \cdot \mathbf{n} = c$.

  Any point having a position vector that satisfies the equation $\mathbf{r} \cdot \mathbf{n} = c$ will lie on the line.

In this chapter, where we are only considering coordinates and vectors in two-dimensional space then:

> $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ (or $\mathbf{r} \cdot \mathbf{n} = c$) is the vector equation of a line passing through the point with position vector **a** and perpendicular to the vector **n**.

$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ (or $\mathbf{r} \cdot \mathbf{n} = c$) is the **scalar product form** of the vector equation of the line. It may also be referred to as the **normal form** of the vector equation of the line. Here the word normal is used because of its geometrical meaning of 'perpendicular' rather than its common meaning of 'usual'.

However, as we will see in the next chapter, if we are considering three-dimensional space, this *scalar product form* defines a plane, not a line.

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EXAMPLE 8

Find **a** the vector equation in scalar product form,

and **b** the cartesian equation,

of a line perpendicular to $5\mathbf{i} - \mathbf{j}$ and passing through point A, position vector $2\mathbf{i} + 3\mathbf{j}$.

Solution

a The vector equation of a line passing through the point with position vector $\mathbf{a}$ and perpendicular to the vector $\mathbf{n}$ is:

$$\mathbf{r}\,.\,\mathbf{n} = \mathbf{a}\,.\,\mathbf{n}.$$

Thus the vector equation of a line passing through the point with a position vector of $2\mathbf{i} + 3\mathbf{j}$ and perpendicular to $5\mathbf{i} - \mathbf{j}$ is:

$$\begin{aligned} \mathbf{r}\,.\,(5\mathbf{i} - \mathbf{j}) &= (2\mathbf{i} + 3\mathbf{j})\,.\,(5\mathbf{i} - \mathbf{j}) \\ &= (2)(5) + (3)(-1) \\ &= 7 \end{aligned}$$

The vector equation of a line perpendicular to $5\mathbf{i} - \mathbf{j}$ and passing through point A, position vector $2\mathbf{i} + 3\mathbf{j}$, is $\mathbf{r}\,.\,(5\mathbf{i} - \mathbf{j}) = 7$.

b If a general point on the line has position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ then:

$$\begin{aligned} (x\mathbf{i} + y\mathbf{j})\,.\,(5\mathbf{i} - \mathbf{j}) &= 7 \\ \text{Thus} \quad (x)(5) + (y)(-1) &= 7 \\ \text{i.e.} \quad 5x - y &= 7 \end{aligned}$$

The cartesian equation of a line perpendicular to $5\mathbf{i} - \mathbf{j}$ and passing through point A, position vector $2\mathbf{i} + 3\mathbf{j}$, is $5x - y = 7$ (i.e. $y = 5x - 7$).

EXAMPLE 9

Find the cartesian equation of the line perpendicular to the vector $4\mathbf{i} - 3\mathbf{j}$ and passing through point A (1, 4).

Solution

The normal form of the vector equation is:

$$\begin{aligned} \mathbf{r}\,.\,(4\mathbf{i} - 3\mathbf{j}) &= (\mathbf{i} + 4\mathbf{j})\,.\,(4\mathbf{i} - 3\mathbf{j}) \\ &= (1)(4) + (4)(-3) \\ &= -8 \end{aligned}$$

Writing $\mathbf{r}$ as $x\mathbf{i} + y\mathbf{j}$ we have

$$\begin{aligned} (x\mathbf{i} + y\mathbf{j})\,.\,(4\mathbf{i} - 3\mathbf{j}) &= -8 \\ (x)(4) + (y)(-3) &= -8 \\ 4x - 3y &= -8 \end{aligned}$$

The required cartesian equation is $4x - 3y = -8$.

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Exercise 4D

1 Find the normal form (i.e. scalar product form) of the vector equation of the line perpendicular to $3\mathbf{i} + 4\mathbf{j}$ and passing through point A, position vector $2\mathbf{i} + 3\mathbf{j}$.

2 Find the normal form (i.e scalar product form) of the vector equation of the line perpendicular to $5\mathbf{i} - \mathbf{j}$ and passing through point A, position vector $-\mathbf{i} + 7\mathbf{j}$.

3 For each of the points A to F given below state whether or not the point lies on the line $\mathbf{r}\cdot(\mathbf{i} + 2\mathbf{j}) = 12$.

Point A, position vector $6\mathbf{j}$.
Point B, position vector $6\mathbf{i} + 3\mathbf{j}$.
Point C, position vector $10\mathbf{i}$.
Point D, position vector $3\mathbf{i} + 6\mathbf{j}$.
Point E, position vector $-4\mathbf{i} + 8\mathbf{j}$.
Point F, position vector $14\mathbf{i} - \mathbf{j}$.

4 Each of the points U to Z given below lies on the line with vector equation:

$$\mathbf{r}\cdot\begin{pmatrix}2\\3\end{pmatrix} = 10.$$

Point U, position vector $u\mathbf{i} + 2\mathbf{j}$.
Point V, position vector $-10\mathbf{i} + v\mathbf{j}$.
Point W, position vector $w\mathbf{i} - 4\mathbf{j}$.
Point X, position vector $x\mathbf{i} - 2\mathbf{j}$.
Point Y, position vector $5\mathbf{i} + y\mathbf{j}$.
Point Z, position vector $z\mathbf{i} + 6\mathbf{j}$.

Determine u, v, w, x, y and z.

5 A line is perpendicular to the vector $5\mathbf{i} + 2\mathbf{j}$ and passes through point A, position vector $\mathbf{i} + \mathbf{j}$.

Find **a** the vector equation of the line in scalar product form,

b the cartesian equation of the line.

6 A line is perpendicular to the vector $2\mathbf{i} + 5\mathbf{j}$ and passes through point A, position vector $2\mathbf{i} - \mathbf{j}$.

Find **a** the vector equation of the line in scalar product form,

b the cartesian equation of the line.

7 Prove that $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} - 4\mathbf{j})$ and $\mathbf{r}\cdot(8\mathbf{i} + 2\mathbf{j}) = 5$ are parallel lines.

8 Find the cartesian equation of the line perpendicular to the vector $8\mathbf{i} + 5\mathbf{j}$ and passing through the point $(-1, 3)$.

9 Prove that lines L_1 and L_2 are perpendicular given that:

L_1 has equation: $\mathbf{r} = 5\mathbf{i} + 2\mathbf{j} + \lambda(3\mathbf{i} - 2\mathbf{j})$.

L_2 has equation: $\mathbf{r}\cdot(6\mathbf{i} - 4\mathbf{j}) = -4$.

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Vector equations of curves in the x–y plane

Earlier in this chapter we saw that the **vector equation** $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j})$

leads to the **parametric equations** $\begin{cases} x = 2 + 2\lambda \\ y = 2 - \lambda \end{cases}$

and, by eliminating λ, the **cartesian equation** $y = -\frac{1}{2}x + 3,$

the vector equation, the parametric equation and the cartesian equation all representing the same straight line.

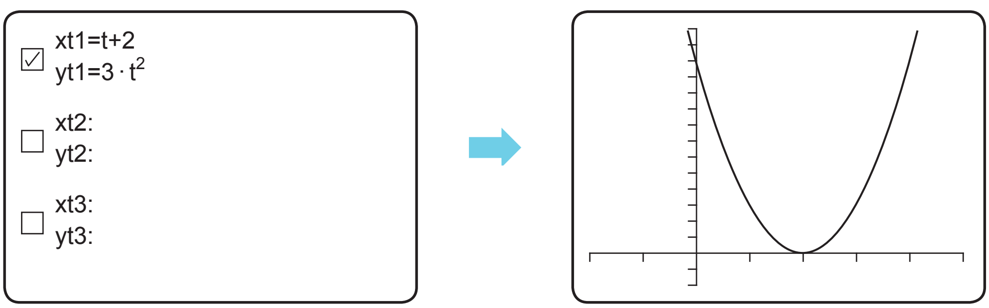
Curves can be similarly represented as vector equations and parametric equations, and, by eliminating the parameter, the cartesian equations we are already familiar with.

For example, the vector equation $\mathbf{r} = (t + 2)\mathbf{i} + 3t^2\mathbf{j}$

leads to the parametric equations $\begin{cases} x = t + 2 \\ y = 3t^2 \end{cases}$

and, by eliminating t, the cartesian equation $y = 3(x - 2)^2.$

Some calculators can display the graphs of functions defined parametrically.

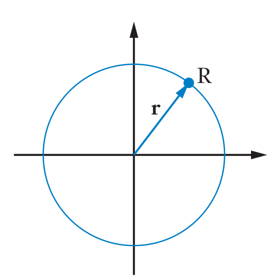


Vector equations of circles in the x–y plane

With a bit of thought, and especially considering the work of an earlier chapter with regard to regions in the complex plane, you should be able to predict the form of the vector equation of a circle.

The vector equation of a circle will be a rule involving $\mathbf{r}$, the position vector of a point on the circle, that will be true for the position vectors of all points lying on the circle, and not true for the position vectors of any points not lying on the circle.

Consider some general point R lying on a circle, centre at the origin and radius a, and let the position vector of R be $\mathbf{r}$. For R to lie on the circle it must be the case that



$$\boxed{|\mathbf{r}| = a}$$

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The vector equation $|\mathbf{r}| = a$ is the **vector equation** of the circle centre (0, 0), radius a.

If the point R has the general cartesian coordinates (x, y) then $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

Thus $\quad |x\mathbf{i} + y\mathbf{j}| = a, \quad$ i.e. $\quad \boxed{x^2 + y^2 = a^2}$

This is the **cartesian equation** of a circle centre (0, 0) and radius a.

Note: We can write the equation of a circle centre at (0, 0) and radius a in parametric form as:

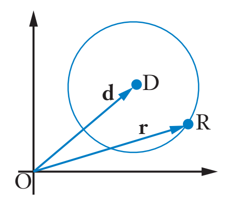
$$\begin{cases} x = a\cos\theta \\ y = a\sin\theta \end{cases}.$$

The reader should confirm that using the trigonometric identity

$$\sin^2\theta + \cos^2\theta = 1$$

to eliminate the parameter θ, gives the cartesian equation $x^2 + y^2 = a^2$.

Consider a circle, again with radius a, but now with its centre not at (0, 0), but instead at the point with position vector $\mathbf{d}$ $(= p\mathbf{i} + q\mathbf{j})$.



Consider some general point R, position vector $\mathbf{r}$, lying on the circle. We require a rule that will be true for the position vectors of all points lying on the circle and not true for points not lying on the circle.

If D is the centre of the circle then, for R to lie on the circle, we must have $|\overrightarrow{\mathrm{DR}}| = a$.

i.e. $\quad \boxed{|\mathbf{r} - \mathbf{d}| = a}$

This is the **vector equation** of a circle of radius a and with its centre at the point with position vector $\mathbf{d}$.

If the point R has the general cartesian coordinates (x, y) then $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

If the centre of the circle has coordinates (p, q) then $\mathbf{d} = p\mathbf{i} + q\mathbf{j}$.

Thus
$$|(x\mathbf{i} + y\mathbf{j}) - (p\mathbf{i} + q\mathbf{j})| = a$$
$$|(x - p)\mathbf{i} + (y - q)\mathbf{j}| = a$$

i.e. $\quad \boxed{(x-p)^2 + (y-q)^2 = a^2}$

This is the **cartesian equation** of a circle centre (p, q) and radius a (as the *Preliminary work* section, and chapter two, reminded us).

Note: We can write the equation of a circle centre at (p, q) and radius a in parametric form as:

$$\begin{cases} x = p + a\cos\theta \\ y = q + a\sin\theta \end{cases}$$

Again the reader should confirm that using the trigonometric identity

$$\sin^2\theta + \cos^2\theta = 1$$

gives the appropriate cartesian equation.

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Remember: If we expand

$$\begin{aligned} (x-p)^2 + (y-q)^2 &= a^2 \text{ we obtain:} \\ x^2 - 2px + p^2 + y^2 - 2qy + q^2 &= a^2 \\ \text{i.e.} \quad x^2 + y^2 - 2px - 2qy &= a^2 - p^2 - q^2 \\ \text{i.e.} \quad x^2 + y^2 - 2px - 2qy &= \text{(a constant)} \end{aligned}$$

In this expanded form the cartesian equation of a circle is characterised by:

- the coefficient of x^2 being the same as the coefficient of y^2,
- the only terms being those in x^2, y^2, x, y and a constant (and of these any two of the last three could be zero).

EXAMPLE 10

a Find the vector equation of the circle centre (0, 0) and radius 5 units.

b For each of the points A, B and C given below determine whether they lie inside, on or outside the circle centre (0, 0) and radius 5 units.

Point A, position vector $3\mathbf{i} - 4\mathbf{j}$. Point B, position vector $2\mathbf{i} + 3\mathbf{j}$. Point C, position vector $4\mathbf{i} - 7\mathbf{j}$.

Solution

a The circle centre (0, 0) and radius a units has vector equation $|\mathbf{r}| = a$.

Thus the circle centre (0, 0) and radius 5 units has vector equation $|\mathbf{r}| = 5$.

b To lie *on* the circle, $\mathbf{r}$ must be such that $|\mathbf{r}| = 5$.

To lie *inside* the circle, $\mathbf{r}$ must be such that $|\mathbf{r}| < 5$.

To lie *outside* the circle, $\mathbf{r}$ must be such that $|\mathbf{r}| > 5$.

For point A $\quad |\mathbf{r}| = |3\mathbf{i} - 4\mathbf{j}|$
$= 5.$ Thus A lies <u>on</u> the circle.

For point B $\quad |\mathbf{r}| = |2\mathbf{i} + 3\mathbf{j}|$
$= \sqrt{13}$ Thus B lies <u>inside</u> the circle.

For point C $\quad |\mathbf{r}| = |4\mathbf{i} - 7\mathbf{j}|$
$= \sqrt{65}$ Thus C lies <u>outside</u> the circle.

EXAMPLE 11

Find the vector equation of the circle centre C, position vector $2\mathbf{i} + 3\mathbf{j}$, and radius 5 units. Determine whether the point A, position vector $5\mathbf{i} - \mathbf{j}$, lies inside, on or outside the circle.

Solution

The vector equation of a circle of radius a and with its centre at the point with position vector $\mathbf{d}$ is:

$$|\mathbf{r} - \mathbf{d}| = a\,.$$

Thus the vector equation of a circle of radius 5 and with its centre at the point with position vector $2\mathbf{i} + 3\mathbf{j}$ is: $\quad |r - (2\mathbf{i} + 3\mathbf{j})| = 5.$

For point A $\quad |\mathbf{r} - (2\mathbf{i} + 3\mathbf{j})| = |(5\mathbf{i} - \mathbf{j}) - (2\mathbf{i} + 3\mathbf{j})|$
$= |3\mathbf{i} - 4\mathbf{j}|$
$= 5$ Point A lies <u>on</u> the circle.

EXAMPLE 12

(As in the *Preliminary work*, but now with mention of the vector equation.)
Find the centre, radius and vector equation of the circle with cartesian equation

$$x^2 + y^2 + 6y = 10x.$$

Solution

$$\begin{array}{lrcl}
\text{Given} & x^2 + y^2 + 6y & = & 10x \\
\text{i.e.} & x^2 - 10x + y^2 + 6y & = & 0 \\
\text{Create gaps:} & x^2 - 10x \qquad + y^2 + 6y & = & 0 \\
\text{Complete the squares:} & x^2 - 10x + 25 + y^2 + 6y + 9 & = & 0 + 25 + 9 \\
\text{Hence} & (x-5)^2 + (y+3)^2 & = & 34
\end{array}$$

The given circle has its centre at $(5, -3)$ and a radius of $\sqrt{34}$ units.

The vector equation of the circle is $|\mathbf{r} - (5\mathbf{i} - 3\mathbf{j})| = \sqrt{34}$.

EXAMPLE 13

Find the position vectors of the points where the straight line $\mathbf{r} = \begin{pmatrix} -6 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ meets the circle $\left| r - \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right| = 5\sqrt{2}$.

Solution

If point A, position vector $\mathbf{r}_A$, lies on both the line and the circle then

$$\mathbf{r}_A = \begin{pmatrix} -6 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \left| r - \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right| = 5\sqrt{2}.$$

Substituting the first expression into the second gives:

$$\begin{array}{rcl}
\left| \begin{pmatrix} -6 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right| & = & 5\sqrt{2}. \\
\left| \begin{pmatrix} \lambda - 8 \\ 2\lambda - 1 \end{pmatrix} \right| & = & 5\sqrt{2}.
\end{array}$$

Thus $\quad (\lambda - 8)^2 + (2\lambda - 1)^2 = 50$

Solving: $\quad \lambda = 1 \quad$ or $\quad \lambda = 3.$

If $\lambda = 1$, $\quad \mathbf{r}_A = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$

If $\lambda = 3$, $\quad \mathbf{r}_A = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$

solve$((\lambda - 8)^2 + (2\lambda - 1)^2 = 50, \lambda)$
$\{\lambda = 1, \lambda = 3\}$

These are the position vectors of the two points common to the line and circle, i.e. the points where the line meets the circle.

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Exercise 4E

1 Find the cartesian equations for each of the following parametric equations.

a $\begin{cases} x = 4 + t \\ y = 2t \end{cases}$

b $\begin{cases} x = t \\ y = \dfrac{1}{t} \end{cases}$

c $\begin{cases} x = t^2 \\ y = 2t \end{cases}$

d $\begin{cases} x = \sqrt{t - 1} \\ y = t^2 \end{cases}$

2 Find the cartesian equations for each of the following vector equations.

a $\mathbf{r} = (3 - t)\mathbf{i} + (4 + 2t)\mathbf{j}$

b $\mathbf{r} = (t - 1)\mathbf{i} + \dfrac{1}{t}\mathbf{j}$

c $\mathbf{r} = (t - 1)\mathbf{i} + (t^2 + 4)\mathbf{j}$

d $\mathbf{r} = (2 + \cos\theta)\mathbf{i} + (1 + 2\sin\theta)\mathbf{j}$

3 Vector equations of the form $\mathbf{r} = (a\cos\theta)\mathbf{i} + (b\sin\theta)\mathbf{j}$ define **ellipses**.

Determine the parametric equations and the cartesian equation of the ellipse with vector equation $\mathbf{r} = (2\cos\theta)\mathbf{i} + (3\sin\theta)\mathbf{j}$.

Use a graphic calculator capable of displaying the graphs of functions defined parametrically to view the ellipse.

4 Vector equations of the form $\mathbf{r} = (a\sec\theta)\mathbf{i} + (b\tan\theta)\mathbf{j}$ define **hyperbolas**.

Determine the parametric equations and the cartesian equation of the hyperbola with vector equation $\mathbf{r} = (-3\sec\theta)\mathbf{i} + (2\tan\theta)\mathbf{j}$.

Use a graphic calculator capable of displaying the graphs of functions defined parametrically to view the hyperbola.

5 Considering only the x–y plane, which of the following equations represent circles?

A: $\mathbf{r} = 2\mathbf{i}$

B: $|\mathbf{r}| = 6$

C: $\mathbf{r} = 6\mathbf{i} + 3\mathbf{j}$

D: $|\mathbf{r} - (5\mathbf{i} - 4\mathbf{j})| = 24$

E: $x^2 + y^2 + 4x - 8y = 5$

F: $x^2 + 8xy + y^2 + 6y = 34$

6 **a** Find the vector equation of the circle with centre (0, 0) and radius 25 units.

b For each of the points A to D given below, determine whether the point lies inside, on, or outside the circle.

Point A, position vector $19\mathbf{i} - 18\mathbf{j}$.

Point B, position vector $-20\mathbf{i} + 15\mathbf{j}$.

Point C, position vector $14\mathbf{i} + 17\mathbf{j}$.

Point D, position vector $-24\mathbf{i} - 7\mathbf{j}$.

7 Find the cartesian equation of a circle with vector equation $|\mathbf{r}| = 65$.

If each of the following points lie on this circle determine a and b given that a is positive and b is negative.

Point A $(-52, a)$.

Point B $(b, 25)$.

8 Find the vector equation of the circle centre C, position vector $-7\mathbf{i} + 4\mathbf{j}$, and radius $4\sqrt{5}$ units. Determine whether the point A, position vector $\mathbf{i} + 8\mathbf{j}$, lies inside, on or outside the circle.

9 Find the vector equation of each of the following circles.

a Centre $(1, -5)$ and radius 9.

b Centre $(-3, 4)$ and radius 10.

c Centre $(-12, 3)$ and radius $2\sqrt{3}$.

d Centre $(-13, -2)$ and radius 4.

10 Find the cartesian equation of each of the following circles, giving your answers in the form $x^2 + y^2 + dx + ey = c$.

a Centre has position vector $2\mathbf{i} + 3\mathbf{j}$. Radius 5.

b Centre has position vector $-4\mathbf{i} + 2\mathbf{j}$. Radius $\sqrt{7}$.

c Centre has position vector $4\mathbf{i} - 3\mathbf{j}$. Radius 7.

11 Find the radius and position vector of the centre of each of the following circles.

a $|\mathbf{r} - (6\mathbf{i} + 3\mathbf{j})| = 5$

b $|\mathbf{r} - 2\mathbf{i} + 3\mathbf{j}| = 6$

c $|(x - 3)\mathbf{i} + (y + 4)\mathbf{j}| = 3$

d $|\mathbf{r}| = 20$

e $16x^2 + 16y^2 = 25$

f $(x - 2)^2 + (y + 3)^2 = 49$

g $x^2 + y^2 - 6x - 18y + 65 = 0$

h $x^2 + y^2 + 20x - 2y = 20$

12 Find the distance between the centres of the two circles given below:

$$|\mathbf{r} - (\mathbf{i} - \mathbf{j})| = 6 \quad \text{and} \quad |\mathbf{r} - 6\mathbf{i} - 11\mathbf{j}| = 7.$$

13 The circle $|\mathbf{r} - (2\mathbf{i} - 5\mathbf{j})| = 5$ has centre A and the circle $|\mathbf{r} - (5\mathbf{i} + 2\mathbf{j})| = 3$ has centre B. Find the vector equation of the straight line through A and B.

14 Point A is the centre of the circle $|\mathbf{r} - (3\mathbf{i} - 2\mathbf{j})| = 3$ and point B is the centre of the circle $|\mathbf{r} - (9\mathbf{i} + 6\mathbf{j})| = 7$. Find $|\overrightarrow{AB}|$.
Determine whether the circles have two points in common, just one point in common or no points in common and justify your answer.

15 Point A is the centre of the circle $|\mathbf{r} - (3\mathbf{i} - \mathbf{j})| = 3$ and point B is the centre of the circle $|\mathbf{r} - (13\mathbf{i} + \mathbf{j})| = 7$. Find $|\overrightarrow{AB}|$.
Determine whether the circles have two points in common, just one point in common or no points in common and justify your answer.

16 Find the position vectors of the points where the straight line $\mathbf{r} = \begin{pmatrix} -10 \\ 15 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -3 \end{pmatrix}$ meets the circle $\left| r - \begin{pmatrix} -1 \\ 7 \end{pmatrix} \right| = \sqrt{29}$.

17 $\mathbf{r} = 10\mathbf{i} - 9\mathbf{j} + \lambda(4\mathbf{i} - 5\mathbf{j})$ is a tangent to the circle $|\mathbf{r} + 7\mathbf{i} - 2\mathbf{j}| = \sqrt{41}$. Find the position vector of the point of contact.

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Closest approach

If two moving particles, each following straight line paths, are not on 'collision course' there will be a moment in time during the motion when they are closer to each other than at any other time (unless they are both travelling at the same velocity and thus constantly maintain the same distance apart).

The distance of closest approach can be determined in a number of ways, as the next example shows.

(If the closest approach occurs for $t \le 0$, then for $t > 0$, the particles are moving further apart.)

EXAMPLE 14

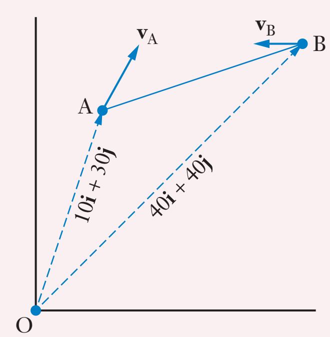
Suppose that at time $t = 0$ two particles, A and B have the following position vectors (**r** metres) and velocity vectors (**v** metres/second):

$$\mathbf{r}_A = 10\mathbf{i} + 30\mathbf{j} \qquad \mathbf{v}_A = 5\mathbf{i} + 9\mathbf{j}$$
$$\mathbf{r}_B = 40\mathbf{i} + 40\mathbf{j} \qquad \mathbf{v}_B = -7\mathbf{i}$$

If the particles continue with these velocities what will be the minimum distance they are apart in the subsequent motion?

Solution

Solution using calculus (or by viewing graph)



At time t, $t > 0$,
$$\begin{aligned}\mathbf{r}_A &= 10\mathbf{i} + 30\mathbf{j} + t(5\mathbf{i} + 9\mathbf{j})\\ &= (10 + 5t)\mathbf{i} + (30 + 9t)\mathbf{j}\end{aligned}$$

and
$$\begin{aligned}\mathbf{r}_B &= 40\mathbf{i} + 40\mathbf{j} + t(-7\mathbf{i})\\ &= (40 - 7t)\mathbf{i} + 40\mathbf{j}\end{aligned}$$

The 'separation vector' from B to A at time t will be:

$$\mathbf{r}_A - \mathbf{r}_B = (-30 + 12t)\mathbf{i} + (-10 + 9t)\mathbf{j}$$

If $|\mathbf{r}_A - \mathbf{r}_B| = d$, $d \ge 0$, then we require d to be a minimum.

But
$$\begin{aligned}d^2 &= (-30 + 12t)^2 + (-10 + 9t)^2\\ &= 225t^2 - 900t + 1000\end{aligned}$$

Viewing the graph of the quadratic function

$$y = 225x^2 - 900x + 1000$$

on a calculator, or using calculus techniques, we can determine that the minimum value of y, i.e. d^2, occurs when x, i.e. t, is equal to 2. With $d \ge 0$ this will also be when d is minimised. For this value of t we have $d_{min} = 10$.

The least distance between the particles in the subsequent motion is 10 metres (and occurs when $t = 2$).

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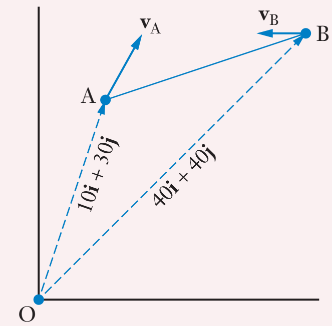
Solution using relative velocities and trigonometry

From the diagram on the right

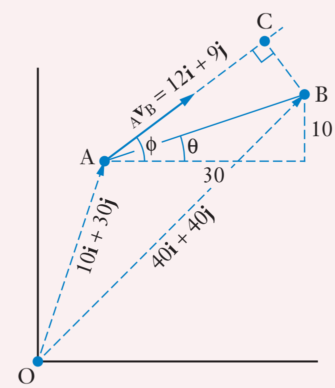


$$\begin{aligned}\overrightarrow{AB} &= -(10\mathbf{i} + 30\mathbf{j}) + (40\mathbf{i} + 40\mathbf{j}) \\ &= 30\mathbf{i} + 10\mathbf{j}\end{aligned}$$

If we subtract $\mathbf{v}_B$ from the velocity of each particle we view the situation as seen by an observer on particle B.

$$\begin{aligned}{}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ &= (5\mathbf{i} + 9\mathbf{j}) - (-7\mathbf{i}) \\ &= 12\mathbf{i} + 9\mathbf{j}\end{aligned}$$

Showing this information in the second diagram we see that the minimum distance between A and B in the subsequent motion is given by CB.



$$\text{Now } \sin(\phi - \theta) = \frac{CB}{AB}$$

$$\begin{aligned}\therefore \quad CB &= AB\sin(\phi - \theta) \\ &= \sqrt{30^2 + 10^2}\,\sin(\phi - \theta)\end{aligned}$$

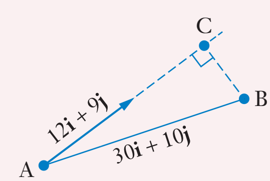
$$\text{But} \quad \tan\theta = \frac{10}{30} \quad \text{and} \quad \tan\phi = \frac{9}{12}$$

By determining θ and ϕ and hence $(\phi - \theta)$ we obtain CB = 10 km.

The least distance between the particles in the subsequent motion is 10 metres.

Solution using relative velocity and scalar product

Consider triangle ABC from the previous diagram.



Suppose the closest approach occurs at time t, $t > 0$.

$$\begin{aligned}\overrightarrow{CB} &= \overrightarrow{CA} + \overrightarrow{AB} \\ &= -[t\,(12\mathbf{i} + 9\mathbf{j})] + 30\mathbf{i} + 10\mathbf{j} \\ &= (30 - 12t)\mathbf{i} + (10 - 9t)\mathbf{j}\end{aligned}$$

$\overrightarrow{CB}$ is perpendicular to $12\mathbf{i} + 9\mathbf{j}$.

$$\begin{aligned}\text{Thus} \quad (12\mathbf{i} + 9\mathbf{j})\,.\,[(30 - 12t)\mathbf{i} + (10 - 9t)\mathbf{j}] &= 0 \\ \therefore \quad 12(30 - 12t) + 9(10 - 9t) &= 0 \\ \text{giving} \quad t &= 2\end{aligned}$$

When $t = 2$, $\overrightarrow{CB} = 6\mathbf{i} - 8\mathbf{j}$ and so $\overrightarrow{CB} = 10$.

The least distance between the particles in the subsequent motion is 10 metres (and occurs when $t = 2$).

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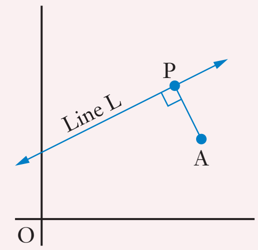
Distance from a point to a line

EXAMPLE 15

Find the perpendicular distance from the point A, position vector $34\mathbf{i} + 12\mathbf{j}$, to the line L, vector equation $\mathbf{r} = 10\mathbf{i} + 14\mathbf{j} + \lambda(5\mathbf{i} + 2\mathbf{j})$.

Solution

Suppose that the perpendicular from A to the line L meets the line at P. Suppose also that at P the value of λ is λ_1.



$$\begin{aligned} \text{Then} \quad \overrightarrow{OP} &= 10\mathbf{i} + 14\mathbf{j} + \lambda_1(5\mathbf{i} + 2\mathbf{j}) \\ \text{Now} \quad \overrightarrow{AP} &= \overrightarrow{AO} + \overrightarrow{OP} \\ &= -(34\mathbf{i} + 12\mathbf{j}) + 10\mathbf{i} + 14\mathbf{j} + \lambda_1(5\mathbf{i} + 2\mathbf{j}) \\ &= (5\lambda_1 - 24)\mathbf{i} + (2\lambda_1 + 2)\mathbf{j} \end{aligned}$$

Line L is parallel to $5\mathbf{i} + 2\mathbf{j}$ and so $(5\mathbf{i} + 2\mathbf{j}) \cdot \overrightarrow{AP} = 0$

$\therefore \quad 5(5\lambda_1 - 24) + 2(2\lambda_1 + 2) = 0$

giving $\quad \lambda_1 = 4$

Hence $\overrightarrow{AP} = -4\mathbf{i} + 10\mathbf{j}$ and so $|\overrightarrow{AP}| = 2\sqrt{29}$.

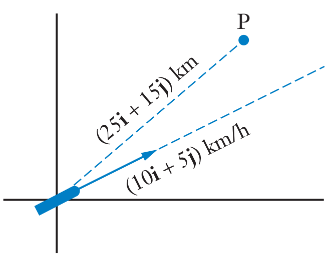
The perpendicular distance from the point A to the line L is $2\sqrt{29}$ units.

Exercise 4F

1 The sketch on the right shows a ship travelling with constant velocity $(10\mathbf{i} + 5\mathbf{j})$ km/h.

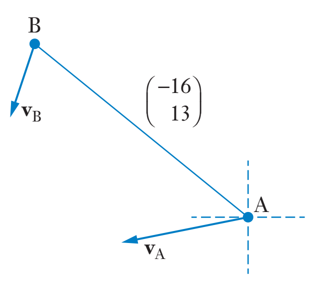
The path of the ship will take it past a fixed offshore drilling platform P.

At 8 a.m. the situation is as shown in the sketch with the platform having a position vector of $(25\mathbf{i} + 15\mathbf{j})$ relative to the ship. How close does the ship come to the drilling platform and when does this closest approach occur?



2 At time $t = 0$ seconds particles A and B are moving with velocities $\begin{pmatrix} -10 \\ -2 \end{pmatrix}$ m/s and $\begin{pmatrix} -2 \\ -6 \end{pmatrix}$ m/s respectively, the position vector of B relative to A being $\begin{pmatrix} -16 \\ 13 \end{pmatrix}$ m.

Assuming A and B maintain these velocities find the least distance of separation between the particles in the subsequent motion and the value of t for which it occurs.



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3 With respect to the location of a mouse, a snake lies in wait at $(5\mathbf{i} + 6\mathbf{j})$ m. The mouse moves in a direction parallel to the vector $\mathbf{i} + 2\mathbf{j}$. The snake, being vectorially astute (!), makes its move when the mouse is at the point on its path that is closest to the snake. If this closest distance is x m then:

If	$x \leq 1$	the snake is certain to catch the mouse.
If	$1 < x < 2$	the snake is more likely to catch the mouse than miss it.
If	$x \geq 2$	the mouse is more likely to escape.

Calculate the value of x and state what is likely to happen.

4 Particles A and B have constant velocities of $(3\mathbf{i} + 4\mathbf{j})$ cm/s and $-3\mathbf{i}$ cm/s respectively. When $t = 0$ seconds B's position relative to A is $(40\mathbf{i} + 5\mathbf{j})$ cm. Find the least distance between the particles during the subsequent motion and the value of t for which it occurs.

5 At 3 a.m. one day the position vectors ($\mathbf{r}$ km) and velocity vectors ($\mathbf{v}$ km/h) of two ships A and B are as follows:

$$\mathbf{r}_A = \begin{pmatrix} 30 \\ 10 \end{pmatrix} \qquad \mathbf{v}_A = \begin{pmatrix} 10 \\ -5 \end{pmatrix}$$

$$\mathbf{r}_B = \begin{pmatrix} 54 \\ -19 \end{pmatrix} \qquad \mathbf{v}_B = \begin{pmatrix} -8 \\ 7 \end{pmatrix}$$

If the ships continue with these velocities what will be the minimum distance they are apart in the subsequent motion?

6 At time $t = 0$ seconds the position vectors ($\mathbf{r}$ m) and velocity vectors ($\mathbf{v}$ m/s) of two particles A and B are as given below:

$$\mathbf{r}_A = 20\mathbf{i} - 10\mathbf{j} \qquad \mathbf{v}_A = 4\mathbf{i} + 5\mathbf{j}$$

$$\mathbf{r}_B = 16\mathbf{i} + 23\mathbf{j} \qquad \mathbf{v}_B = 6\mathbf{i} - 3\mathbf{j}$$

If the particles continue with these velocities what will be the minimum distance they are apart in the subsequent motion?

7 Find the perpendicular distance from the point A, position vector $14\mathbf{i} - 3\mathbf{j}$, to the line L, vector equation $\mathbf{r} = -5\mathbf{i} + 22\mathbf{j} + \lambda(5\mathbf{i} - 2\mathbf{j})$.

8 Find the perpendicular distance from the point A, position vector $\begin{pmatrix} 11 \\ 18 \end{pmatrix}$, to the line L, vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

9 Find the perpendicular distance from the point A, position vector $\begin{pmatrix} -3 \\ 8 \end{pmatrix}$, to the line L, vector equation $\mathbf{r} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -2 \end{pmatrix}$.

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Miscellaneous exercise four

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

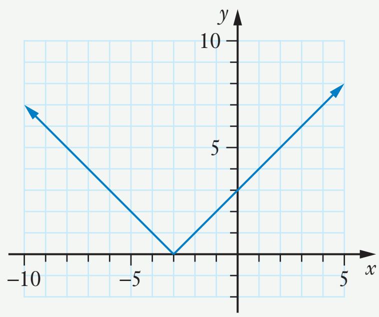
1 Prove that the lines L_1 and L_2 are perpendicular given that

L_1 has equation $\quad \mathbf{r} = 2\mathbf{i} + 7\mathbf{j} + \lambda(10\mathbf{i} + 4\mathbf{j})$

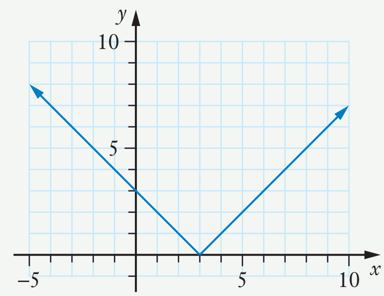
and L_2 has equation $\quad \mathbf{r} = 3\mathbf{i} - 4\mathbf{j} + \mu(-2\mathbf{i} + 5\mathbf{j})$.

2 Write the equations of each of the absolute value functions shown below.

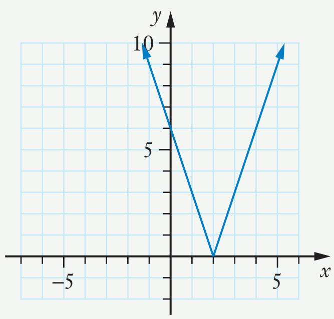
a



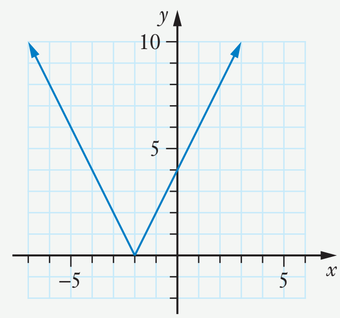
b



c



d



3 Find the radius and the cartesian coordinates of the centre of each of the following circles.

a $|\mathbf{r} - (7\mathbf{i} - \mathbf{j})| = 5$

b $|\mathbf{r} - 7\mathbf{i} - \mathbf{j}| = 6$

c $x^2 + y^2 = 18$

d $(x - 1)^2 + (y + 8)^2 = 75$

e $x^2 + y^2 + 2x = 14y + 50$

f $x^2 + 10x + y^2 = 151 + 14y$

4 The graph of $y = f(x)$ is shown on the right.

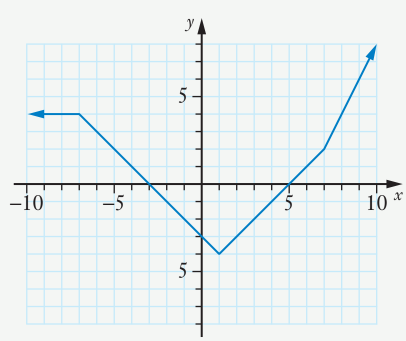
Produce sketch graphs of each of the following:

a $y = -f(x)$

b $y = f(-x)$

c $y = |f(x)|$

d $y = f(|x|)$



5 **a** $3x^3 - 11x^2 + 25x - 25 = (ax - b)(x^2 + cx + 5)$ for real integers a, b and c. Determine a, b and c.

b Showing full algebraic reasoning, find all values of x, real and complex, for which $3x^3 - 11x^2 + 25x - 25 = 0$.

c Check your answers to **b** using a calculator.

6 Given that $f(x) = 1 - \dfrac{1}{\sqrt{4 - x}}$ determine

a $f(-21)$, **b** $f[f(3)]$.

Viewing the graph of $f(x)$ on your calculator if you wish, determine

c the domain of f, **d** the range of f.

e State the domain and range of f^{-1} and find its rule.

7 **a** Express the complex number $z = 2\,\text{cis}\left(\dfrac{\pi}{6}\right)$ in cartesian form, $a + ib$.

b Express the complex number $w = -1 - \sqrt{3}i$ in polar form, $r\,\text{cis}\,\theta$, with $r \geq 0$ and $-\pi < \theta \leq \pi$.

c Express zw in both polar and cartesian form.

d Express $\dfrac{z}{w}$ in both polar and cartesian form.

8 For each of the following determine whether the given straight line cuts the circle in two places, touches it tangentially or does not touch or cut it at all. For those parts in which the line and the circle do meet give the position vector(s) of the point(s) of contact.

a The line $\mathbf{r} = -10\mathbf{i} + 24\mathbf{j} + \lambda(5\mathbf{i} + \mathbf{j})$ and the circle $|\mathbf{r} - (34\mathbf{i} + 12\mathbf{j})| = 2\sqrt{130}$.

b The line $\mathbf{r} = -\mathbf{i} + 5\mathbf{j} + \lambda(3\mathbf{i} - \mathbf{j})$ and the circle $|\mathbf{r} - (3\mathbf{i} + \mathbf{j})| = \sqrt{5}$.

c The line $\mathbf{r} = -\mathbf{i} + 7\mathbf{j} + \lambda(\mathbf{i} + 3\mathbf{j})$ and the circle $|\mathbf{r} - (4\mathbf{i} + 2\mathbf{j})| = 2\sqrt{10}$.

9 Without simply using de Moivre's theorem, prove that if $\text{cis}\,\theta = \cos\theta + i\sin\theta$ then

$$\frac{1}{\text{cis}\,\theta} = \text{cis}(-\theta).$$

10 **a** If $z = \cos\theta + i\sin\theta$ use de Moivre's theorem to prove that:

$$z^k + \frac{1}{z^k} = 2\cos(k\theta)$$

b If $k = 1$ then it follows that $z + \dfrac{1}{z} = 2\cos\theta$

Use this fact and the result from **a** to prove:

i $\cos^3\theta = \dfrac{\cos(3\theta) + 3\cos\theta}{4}$ **ii** $\cos^4\theta = \dfrac{\cos(4\theta) + 4\cos(2\theta) + 3}{8}$

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